Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Approximation Theory 137 (2005) 123–142

Journal of Approximation Theory

www.elsevier.com/locate/jat

Convergence rates of vector cascade algorithms in L_p ☆

Song Li

Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang, 310027, PR China

Received 19 November 2004; accepted 7 July 2005

Communicated by Rong-Qing Jia
Available online 5 October 2005

Abstract

We investigate the solutions of vector refinement equations of the form

$$\varphi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\varphi(M \cdot -\alpha),$$

where the vector of functions $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ is in $(L_p(\mathbb{R}^s))^r$, $1 \leqslant p \leqslant \infty$, $a =: (a(\alpha))_{\alpha \in \mathbb{Z}^s}$ is a finitely supported sequence of $r \times r$ matrices called the refinement mask, and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. Associated with the mask a and M is a linear operator Q_a defined on $(L_p(\mathbb{R}^s))^r$ by $Q_a\psi := \sum_{\beta \in \mathbb{Z}^s} a(\beta)\psi(M \cdot -\beta)$. The iteration scheme $(Q_a^n\psi)_{n=1,2,\ldots}$ is called a cascade algorithm (see [D.R. Chen, R.Q. Jia, S.D. Riemenschneider, Convergence of vector subdivision schemes in Sobolev spaces, Appl. Comput. Harmon. Anal. 12 (2002) 128–149; B. Han, The initial functions in a cascade algorithm, in: D.X. Zhou (Ed.), Proceeding of International Conference of Computational Harmonic Analysis in Hong Kong, 2002; B. Han, R.Q. Jia, Multivariate refinement equations and convergence of subdivision schemes, SIAM J. Math. Anal. 29 (1998) 1177–1199; R.Q. Jia, Subdivision schemes in L_p spaces, Adv. Comput. Math. 3 (1995) 309–341; R.Q. Jia, S.D. Riemenschneider, D.X. Zhou, Vector subdivision schemes and multiple wavelets, Math. Comp. 67 (1998) 1533–1363; S. Li, Characterization of smoothness of multivariate refinable functions and convergence of cascade algorithms associated with nonhomogeneous refinement equations, Adv. Comput. Math. 20 (2004) 311–331; Q. Sun, Convergence and boundedness of cascade algorithm in Besov space and Triebel–Lizorkin space I, Adv. Math. (China) 29 (2000) 507–526]). Cascade algorithm is an important issue to wavelets analysis and computer graphics. Main results of this paper are related to the convergence and convergence rates of vector cascade algorithm in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$. We give some characterizations on convergence of cascade algorithm and also give estimates on convergence rates of this cascade algorithm with M being isotropic dilation matrix. It is well known that smoothness is a very

☆ This project was supported by NSF of China under Grant numbers 10071071 and 10471123.
E-mail address: zsj@zju.edu.cn.

doi:10.1016/j.jat.2005.07.010

important property of a multiple refinable function. A characterization of $L_p(1 \leqslant p \leqslant \infty)$ smoothness of multiple refinable functions is also presented when $M = qI_{s\times s}$, where $I_{s\times s}$ is the $s \times s$ identity matrix, and $q \geqslant 2$ is an integer. In particular, the smoothness results given in [R.Q. Jia, S.D. Riemenschneider, D.X. Zhou, Smoothness of multiple refinable functions and multiple wavelets, SIAM J. Matrix Anal. Appl. 21 (1999) 1–28] is a special case of this paper.

MSC: 42C40; 41A10; 41A15; 41A25

Keywords: Refinement equation; Cascade algorithm; Multiple refinable functions; Convergence rates; Smoothness

1. Introduction

A vector refinement equation is a functional equation of the form

$$\varphi = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\varphi(M\cdot-\alpha), \quad x \in \mathbb{R}^s, \tag{1.1}$$

where $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ is in $(L_p(\mathbb{R}^s))^r$, $1 \leqslant p \leqslant \infty$, a is a finitely supported refinement mask such that each $a(\alpha)$ is an $r \times r$ (complex) matrix and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. Eq. (1.1) is called a homogeneous vector refinement equation. The solutions of Eqs. (1.1) are called multiple refinable functions. It is well known that refinement equations play an important role in wavelet analysis and computer graphics (see [1,4,5,7–9,12,17,19–23,25,26]). Most useful wavelets in applications are generated from refinable functions. The cascade algorithm and smoothness of refinable function are two important issues in wavelets analysis and computer graphics. For example, cascade algorithm can be used to characterize the existence of solution and orthogonality of shifts of solution of Eq. (1.1) and also have strong impact on applications of wavelet to computer graphics. Smooth wavelets are needed in image processing and signal processing. The approximation and smoothness properties of wavelets are determined by the corresponding refinable functions. Therefore, it is very important to investigate the convergence of cascade algorithm and smoothness of refinable functions.

Before proceeding, we introduce some notations. For $1 \leqslant p \leqslant \infty$, by $(L_p(\mathbb{R}^s))^r$ we denote the linear space of all vectors $f = (f_1, \ldots, f_r)^T$ such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\sum_{j=1}^{r} \int_{\mathbb{R}^s} |f_j|^p dx\right)^{1/p}, \quad 1 \leqslant p < \infty,$$

and $\|f\|_\infty$ is the essential supremum of $\max_{1\leqslant j\leqslant r} |f_j|$ on $\mathbb{R}^s$. When $1 \leqslant p \leqslant \infty$, $\|\cdot\|_p$ is a norm and, equipped with this norm, $(L_p(\mathbb{R}^s))^r$ is a Banach space.

The Fourier transform of a vector of functions in $(L_1(\mathbb{R}^s))^r$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\xi} dx, \quad \xi \in \mathbb{R}^s, \tag{1.2}$$

where $x \cdot \xi$ denotes the inner produce of two vectors x and ξ in $\mathbb{R}^s$.

The study of smoothness of φ is related to properties of shift-invariant spaces. Suppose $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ is an $r \times 1$ vector of compactly supported functions in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$. We use $S(\varphi)$ to denote the shift-invariant space generated from φ, which is the linear space of functions of the form

$$\sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^s} b_j(\alpha)\varphi_j(\cdot - \alpha),$$

where $b_1, \ldots, b_r \in \ell(\mathbb{Z}^s)$, the linear space of all sequences on $\mathbb{Z}^s$. The shifts of compactly supported functions $\varphi_1, \ldots, \varphi_r \in L_p(\mathbb{R}^s)(1 \leqslant p \leqslant \infty)$ are said to be L_p-stable if there exist two positive constants C_1 and C_2 such that, for arbitrary $b_1, \ldots, b_r \in \ell_p(\mathbb{Z}^s)$,

$$C_1 \sum_{j=1}^{r} \|b_j\|_p \leqslant \left\| \sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^s} b_j(\alpha)\varphi_j(\cdot - \alpha) \right\|_p \leqslant C_2 \sum_{j=1}^{r} \|b_j\|_p,$$

where $\ell_p(\mathbb{Z}^s)$ denotes the linear space of all sequence c for which $\|c\|_p < \infty$, the ℓ_p-norm of c is defined by

$$\|c\|_p := \left(\sum_{\alpha \in \mathbb{Z}^s} |c(\alpha)|^p \right)^{1/p}, \quad 1 \leqslant p < \infty,$$

and $\|c\|_\infty$ is the supremum of $|c|$ on $\mathbb{Z}^s$. Clearly, $\|\cdot\|_p$ is a norm for $1 \leqslant p \leqslant \infty$. It was proved in [16] that the shifts of $\varphi_1, \ldots, \varphi_r$ are L_p-stable if and only if, for any $\xi \in \mathbb{R}^s$, the sequences $(\hat{\varphi}_j(\xi + 2\beta\pi))_{\beta \in \mathbb{Z}^s}$, $j = 1, \ldots, r$ are linear independent. In [10], Jia obtained a similar characterization for L_p-stability of the shifts of a finite number of compactly supported distributions when $0 < p < 1$.

We denote the set of all positive integers by $\mathbb{N}$, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A multi-index is an s-tuple $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$. The length of μ is $|\mu| := \mu_1 + \cdots + \mu_s$. The partial derivative of a differentiable function f with respect to the jth coordinate is denoted by $D_j f$, $j = 1, \ldots, s$, and for $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, D^μ is the partial differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$. For $1 \leqslant p \leqslant \infty$ and an integer $k \geqslant 0$, we use $W_p^k(\mathbb{R}^s)$ to denote the Sobolev space that consists of all distributions f such that $D^\mu f \in L_p(\mathbb{R}^s)$ for all multi-indices μ, with $|\mu| \leqslant k$. The Sobolev norm is defined by

$$\|f\|_{W_p^k(\mathbb{R}^s)} := \sum_{|\mu| \leqslant k} \left\| D^\mu f \right\|_p .$$

With above norm, $W_p^k(\mathbb{R}^s)$ becomes a Banach space.

We use $(W_p^k(\mathbb{R}^s))^r$ to denote the space that consists of all vector of functions $f = (f_1, \ldots, f_r)^T$ such that $f_j \in W_p^k(\mathbb{R}^s)$ for $j = 1, 2, \ldots, r$. The norm on $(W_p^k(\mathbb{R}^s))^r$ is denoted by

$$\|f\|_{(W_p^k(\mathbb{R}^s))^r} := \left(\sum_{j=1}^{r} \|f_j\|_{W_p^k(\mathbb{R}^s)}^p \right)^{1/p}.$$

If $\varphi = (\varphi_1, \ldots, \varphi)^T$ is a compactly supported solution of (1.1) in $(W_p^k(\mathbb{R}^s))^r$ for $1 \leqslant p \leqslant \infty$. Taking the Fourier transform of both sides of (1.1), we obtain

$$\hat{\varphi}(\xi) = H((M^T)^{-1}\xi)\hat{\varphi}((M^T)^{-1}\xi), \quad \xi \in \mathbb{R}^s, \tag{1.3}$$

where M^T denotes the transpose of M, and

$$H(\xi) := \frac{1}{|det\ M|} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^s. \tag{1.4}$$

Evidently, $H(\xi)$ is 2π-periodic. If $\hat{\varphi}(0) \neq 0$, then $\hat{\varphi}(0)$ is an eigenvector of the matrix $H(0)$ corresponding to eigenvalue 1. It was also proved in [1] that if φ satisfies $\hat{\varphi}(0) \neq 0$ and span $\{\hat{\varphi}(2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$, then 1 is a simple eigenvalue of $H(0)$ and other eigenvalues of $H(0)$ are less than $(\rho(M))^{-k}$ in modulus, where $\rho(M)$ denote the spectral radius of dilation matrix M. These conditions are called Eigenvalue Condition. Under above conditions on $H(0)$, refinement equation (1.1) has a unique compactly supported distributional solution φ up to a constant factor. Let φ be a solution of vector refinement equation (1.1) in $(W_p^k(\mathbb{R}^s))^r$ for $1 \leqslant p \leqslant \infty$, if the shifts of φ is stable, then $H(0)$ satisfies Eigenvalue condition.

In this paper we assume that $H(0)$ satisfies Eigenvalue condition. Thus, there is a nonsingular matrix V so that $VH(0)V^{-1}$ has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix}, \tag{1.5}$$

where Λ is an $(r-1)\times(r-1)$ matrix that satisfies $\lim_{n\to\infty} \Lambda^n = 0$. Define $b(\alpha) = Va(\alpha)V^{-1}$, then $\psi = V\varphi$ satisfies the refinement equation

$$\psi = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha)\psi(M \cdot -\alpha), \quad x \in \mathbb{R}^s, \tag{1.6}$$

where φ is a solution of (1.1). Therefore, we may assume that the $r \times r$ matrix $H(0)$ has the form (1.5), without losing anything.

For $j = 1, 2, \ldots, r$, we use e_j to denote the jth column of the $r \times r$ identity matrix. It is easily seen that

$$e_1^T H(0) = e_1^T. \tag{1.7}$$

In order to study L_p-solutions of the refinement equation (1.1) we shall employ the following iteration scheme. Let Q_a be the linear operator on $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ given by

$$Q_a f := \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) f(M \cdot -\alpha), \qquad f \in (L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty). \tag{1.8}$$

Let φ_0 be an $r \times 1$ initial vector of functions in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$. For $n = 1, 2, \ldots$, let $\varphi_n := Q_a^n \varphi_0$. If $\{\varphi_n\}_{n=1,2,\ldots}$ converges to some φ in the $(L_p(\mathbb{R}^s))^r$ space for $1 \leqslant p \leqslant \infty$, then the limit φ is a solution of (1.1) in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$. Iterating (1.8) n times gives

$$Q_a^n \varphi_0 = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) \varphi_0(M^n \cdot -\alpha), \quad n = 1, 2, \ldots, \tag{1.9}$$

where for $n = 1, 2, \ldots, a_1 = a$ and a_n is defined by following iterative relations:

$$a_n(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_{n-1}(\beta) a(\alpha - M\beta), \quad \alpha \in \mathbb{Z}^s. \tag{1.10}$$

The iteration scheme $\{Q_a^n \varphi_0\}_{n=1,2,\ldots}$ is called cascade algorithm. The cascade algorithms play an important role in computer graphics and wavelet analysis. Therefore, there are many papers to be devoted to studying convergence of cascade algorithms mentioned above (see [1,5,6,9,12,17,27,28]). Let φ be the unique compactly supported distributional solution of refinement equation (1.1) subject to the condition $e_1^T\hat{\varphi}(0)=1$. Such a solution is called the normalized solution of refinement equation (1.1). It was proved in [21] that if $\lim_{n\to\infty}\|Q_a^n\varphi_0-\varphi\|_p=0$, then φ_0 must satisfy Strang-Fix conditions of order 1. A typical choice of initial function vector is given by

$$\varphi_0=e_1^T\prod_{j=1}^{s}\chi(x_j),\qquad x=(x_1,\ldots,x_s)\in\mathbb{R}^s,$$

where $\chi(t)$ is the characteristic function of the interval $[0,1)$. We will prove that if $\lim_{n\to\infty}\|Q_a^n\varphi_0-\varphi\|_p=0$ for this initial function vector φ_0, then $\lim_{n\to\infty}\|Q_a^n\psi-\varphi\|_p=0$ for any vector of compactly supported functions ψ in $(L_p(\mathbb{R}^s))^r(1\leqslant p\leqslant\infty)$ satisfying Strang-Fix conditions of order 1. Reader is referred to [14] for some related discussions on Strang-Fix conditions. However, the sequence $\{Q_a^n\varphi_0\}_{n=1,2,\ldots}$ may diverge for this initial function vector, even if the normalized solution φ lies in $(L_p(\mathbb{R}^s))^r(1\leqslant p\leqslant\infty)$. Thus, it is necessary to find appropriate conditions on initial function vector φ_0 such that

$$\lim_{n\to\infty}\|Q_a^n\varphi_0-\varphi\|_p=0.$$

When $r=1$, Jia [12] obtained a characterization on initial vector of functions φ_0 to ensure $\lim_{n\to\infty}\|Q_a^n\varphi_0-\varphi\|_p=0$. When $r>1$, $M=2$ and $1<p<\infty$, Sun also established a similar characterization in [29] with a different method. Let us simply review Jia's results.

Theorem A (*Jia [12]*). *Let φ be the normalized solution of the refinement equation* (1.1) *with $r=1$ and in $L_p(\mathbb{R}^s)(1\leqslant p\leqslant\infty)$. Let ψ be a compactly supported function in $L_p(\mathbb{R}^s)$ satisfying Strang-Fix conditions of order* 1. *If $K(\varphi)\subseteq K(\psi)$, then*

$$\lim_{n\to\infty}\left\|Q_a^n\psi-\varphi\right\|_p=0,$$

where $K(\varphi)=\left\{c(\alpha)\in\ell(\mathbb{Z}^s);\ \sum_{\alpha\in\mathbb{Z}^s}c(\alpha)\varphi(\cdot-\alpha)=0\right\}$.

We point out that the shifts of φ is linear independent if $K(\varphi)=\{0\}$. Jia also established estimates for the convergence rate of cascade algorithm with dilation matrix M being isotropic. A dilation matrix M is isotropic if M is similar to a diagonal matrix $diag(\sigma_1,\ldots,\sigma_s)$ with $|\sigma_1|=\cdots=|\sigma_s|$.

Theorem B (*Jia [12]*). *Let φ be the normalized solution of the refinement equation* (1.1) *with $r=1$ and dilation matrix M being isotropic. Suppose $\varphi\in Lip(v,L_p(\mathbb{R}^s))$, where $v>0$ and $1\leqslant p\leqslant\infty$. Suppose k is an integer such that $k-1<v\leqslant k$. Let ψ be a compactly supported function in $L_p(\mathbb{R}^s)$ satisfying Strang-Fix conditions of order k. If $K(\varphi)\subseteq K(\psi)$, and if $D^\mu\hat{\varphi}(0)=D^\mu\hat{\psi}(0)$ for all $|\mu|<k$, then there exists a positive constant C such that*

$$\left\|Q_a^n\psi-\varphi\right\|_p\leqslant C(m^{-1/s})^{nv}\quad\forall n\in\mathbb{N}.$$

We remark that a result similar to Theorem B was also established in [29] for the case $r>1, M=2$ and $1<p<\infty$. In particular, Theorem B confirms a conjecture of Ron on convergence of cascade algorithms (see [26]).

In Section 2, we will investigate the convergence and convergence rates of cascade algorithm associated with vector refinement equations (1.1) in $(L_p(\mathbb{R}^s))^r (1\leqslant p\leqslant\infty)$. We will extend above theorems to vector refinement equation (1.1). This goal will be achieved by employing some approaches used in [1,13,12,14,21].

We use the generalized Lipschitz space to measure smoothness of a given function. For $\lambda\in\mathbb{R}^s$, the difference operator ∇_λ is defined by

$$\nabla_\lambda f := f - f(\cdot-\lambda).$$

The difference operator ∇_j on $\ell_0(\mathbb{Z}^s)$ denoted by

$$\nabla_j a := a - a(\cdot - e_j).$$

The modulus of continuity of a function f in $L_p(\mathbb{R}^s)(1\leqslant p\leqslant\infty)$ is defined by

$$\omega(f,h)_p := \sup_{|\lambda|\leqslant h} \|\nabla_\lambda f\|_p, \quad h\geqslant 0.$$

For $1\leqslant p\leqslant\infty$ and $0 < v\leqslant 1$, the Lipschitz space Lip$(v, L_p(\mathbb{R}^s))$ consists of those functions $f\in L_p(\mathbb{R}^s)$ for which

$$\omega(f,h)_p\leqslant Ch^v \quad \forall h>0,$$

where C is a positive constant independent of h. When $v>1$, we write $v=l+\eta$, where l is an integer and $0<\eta\leqslant 1$. The Lipschitz space Lip$(v, L_p(\mathbb{R}^s))$ consists of those functions $f\in L_p(\mathbb{R}^s)$ for which $D^\mu f\in Lip(\eta, L_p(\mathbb{R}^s))$ for all multi-indices μ with $|\mu|=l$.

We use $(Lip(v, L_p(\mathbb{R}^s)))^r$ to denote the linear space of all vectors $f=(f_1,\ldots,f_r)^T$ such that $f_1,\ldots,f_r\in Lip(v, L_p(\mathbb{R}^s))$. Let k be a positive integer. The kth modulus of smoothness of $f\in L_p(\mathbb{R}^s)(1\leqslant p\leqslant\infty)$ is defined by

$$\omega_k(f,h)_p := \sup_{|\lambda|\leqslant h} \|\nabla_\lambda^k f\|_p, \quad h\geqslant 0.$$

For $v>0$, let k be an integer greater than v. The generalized Lipschitz space Lip*$(v, L_p(\mathbb{R}^s))$ consists of those functions $f\in L_p(\mathbb{R}^s)$ for which

$$\omega_k(f,h)_p\leqslant Ch^v \quad \forall h>0,$$

where C is a positive constant independent of h.

By $(Lip^*(v, L_p(\mathbb{R}^s)))^r$ we denote the linear space of all vectors $f=(f_1,\ldots,f_r)^T$ such that $f_1,\ldots,f_r\in Lip^*(v, L_p(\mathbb{R}^s))$. The optimal smoothness of a vector $f\in(L_p(\mathbb{R}^s))^r$ in the L_p-norm is described by its critical exponent $v_p(f)$ defined by

$$v_p(f) := \sup\left\{v : f\in (Lip^*(v, L_p(\mathbb{R}^s)))^r\right\}.$$

Our another concern is the L_p smoothness of solution of refinement equation (1.1) with $M=qI_{s\times s}$, where $q\geqslant 2$ is an integer. When $s=1$ and $M=2$, Jia et al. [18] gave a characterization of smoothness of multiple refinable function in $(L_p(\mathbb{R}))^r(1\leqslant p\leqslant\infty)$. When $r=1$ and dilation matrix is isotropic, Jia [11] obtained a characterization of L_2 smoothness of multi-variate refinable function. In [22], author obtained a similar characterization of L_2 smoothness of multiple refinable functions with M being isotropic matrix. In Section 3, we will give a characterization of $L_p(1\leqslant p\leqslant\infty)$ smoothness of solutions of vector refinement equation (1.1) with $M=qI_{s\times s}$. In particular, the smoothness results given in [18] are the special case of this paper.

In Section 3, we will establish lower bounds of $v_p(\varphi)$, when φ is a L_p solution of Eq. (1.1) for $1 \leqslant p \leqslant \infty$. We will also show that under some appropriate conditions on φ, the lower bound is optimal.

2. Convergence rates of cascade algorithm in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$

In this section we investigate the convergence and convergence rates of cascade algorithm associated with refinement equation (1.1) by using of some ideas of [1,13,12,14,21]. Let $(\ell_0(\mathbb{Z}^s))^{r \times d}$ and $(\ell(\mathbb{Z}^s))^{r \times d}$ denote the linear spaces of all finitely supported sequences of $r \times d$ matrices and all sequences of $r \times d$ matrices on $\mathbb{Z}^s$, respectively. We introduce a bilinear form on a pair of linear space $(\ell_0(\mathbb{Z}^s))^{r \times 1}$ and $(\ell(\mathbb{Z}^s))^{1 \times r}$ as follows:

$$< u, v > := \sum_{\alpha \in \mathbb{Z}^s} u(-\alpha) v(\alpha), \quad u \in (\ell(\mathbb{Z}^s))^{1 \times r}, \;\; v \in (\ell_0(\mathbb{Z}^s))^{r \times 1}.$$

It is known that $(\ell(\mathbb{Z}^s))^{1 \times r}$ is the algebraic dual of $(\ell_0(\mathbb{Z}^s))^{r \times 1}$ with respect to this bilinear form.

If $< u, v >= 0$, then u and v are called orthogonal and we write $u \perp v$. The annihilator of a linear subspace V of $(\ell_0(\mathbb{Z}^s))^{r \times 1}$ is defined by

$$V^{\perp} := \left\{ u \in (\ell(\mathbb{Z}^s))^{1 \times r}; < u, v >= 0 \quad \forall v \in V \right\}.$$

We need the following Lemma 2.1 which was established in [14].

Lemma 2.1. *If V is a linear subspace of* $(\ell_0(\mathbb{Z}^s))^{r \times 1}$, *then* $V = (V^{\perp})^{\perp}$.

Let $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ be an $r \times 1$ vector of compactly supported functions on $\mathbb{R}^s$, we use $K(\varphi)$ to denote the linear space of all sequence $c = (c_1, \ldots, c_r)$ on $\mathbb{Z}^s$ such that

$$\sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^s} c_j(\alpha) \varphi_j(\cdot - \alpha) = 0.$$

We say that the shifts of $\varphi_1, \ldots, \varphi_r$ are linear independent if $K(\varphi) = \{0\}$. It was proved in [16] that the shifts of $\varphi_1, \ldots, \varphi_r$ are linear independent if and only if for any $\xi \in \mathbb{C}^r$,

$$span \left\{ \hat{\varphi}(\xi + 2\pi\beta) : \beta \in \mathbb{Z}^s \right\} = \mathbb{C}^r.$$

Hence, linear independent implies stability.

The following lemma extends Lemma 2.1 of [12] to the case $r > 1$ which is also of independent interest.

Lemma 2.2. *Let* φ *and* ψ *be vectors of compactly supported functions in* $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$. *Suppose* $K(\varphi) \subseteq K(\psi)$. *Then there exists a positive constant C such that*

$$\left\| \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) \psi(\cdot - \alpha) \right\|_p \leqslant C \left\| \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) \varphi(\cdot - \alpha) \right\|_p \qquad \forall b \in (\ell_p(\mathbb{Z}^s))^{1 \times r}.$$

Proof. Note that $S(\varphi)$ denotes the shift-invariant space generated by φ. In other words,

$$S(\varphi) = \left\{ \sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^s} u_j(\alpha)\varphi_j(\cdot - \alpha) : u_1, \ldots, u_r \in \ell(\mathbb{Z}^s) \right\}.$$

Since $\varphi_1, \ldots, \varphi_r$ are compactly supported. The restriction of $S(\varphi)$ to the cube $[0, 1]^s$ is finite dimensional. Thus, we can find functions $\phi_1, \ldots, \phi_d \in L_p(\mathbb{R}^s)(1 \leqslant p \leqslant \infty)$ with support in $[0, 1]^s$ such that $\phi_1|_{[0,1]^s}, \ldots, \phi_d|_{[0,1]^s}$ form a basis for $S(\varphi)|_{[0,1]^s}$. It follows that, for φ_j, $j = 1, 2, \ldots, r$, there exist sequences $c_{j1}, c_{j2}, \ldots, c_{jd} \in \ell_0(\mathbb{Z}^s)$ such that

$$\varphi_j = \sum_{l=1}^{d} \sum_{\alpha \in \mathbb{Z}^s} c_{jl}(\alpha)\phi_l(\cdot - \alpha).$$

For $b \in (\ell_p(\mathbb{Z}^s))^{1\times r}$, we have

$$\sum_{\alpha \in \mathbb{Z}^s} b(\alpha)\varphi(\cdot - \alpha) = \sum_{\gamma \in \mathbb{Z}^s} (b * c)(\gamma)\phi(\cdot - \gamma), \tag{2.1}$$

where $c = \left(c_{jl}\right)_{1 \leqslant j \leqslant r, 1 \leqslant l \leqslant d} \in (\ell_0(\mathbb{Z}^s))^{r\times d}$, $\phi = (\phi_1, \ldots, \phi_d)^T$, and $c * b$ denotes the convolution of b and c, given by

$$b * c(\alpha) = \sum_{\beta \in \mathbb{Z}^s} b(\alpha - \beta)c(\beta).$$

Since the functions $\phi_1, \ldots, \phi$ are supported in $[0, 1]^s$ and are linearly independent, we know that there exists a constant C_1 such that

$$\left\| \sum_{\alpha \in \mathbb{Z}^s} b(\alpha)\varphi(\cdot - \alpha) \right\|_p \geqslant C_1 \left\| b * c_l \right\|_p, \quad l = 1, 2, \ldots, d, \tag{2.2}$$

where $c_l \in (\ell_0(\mathbb{Z}^s))^{r\times 1}$ is the lth column of c.

Let V_φ be the linear span of (multi-integer) shifts of $c_1, \ldots, c_d$. It follows from (2.1) that $b \in K(\varphi)$ if and only if b is orthogonal to shifts of $c_1, \ldots, c_d$, i.e., $< b, c_l(\cdot - \beta) >= 0$, $l = 1, 2, \ldots, d, \forall \beta \in \mathbb{Z}^s$. Therefore, we know that $K(\varphi) = V_\varphi^\perp$.

Similarly, we can find linear independent functions $\psi'_1, \ldots, \psi'_n$ in $L_p(\mathbb{R}^s)$ supported in $[0, 1]^s$ such that

$$\psi_j = \sum_{l=1}^{n} \sum_{\alpha \in \mathbb{Z}^s} h_{jl}(\alpha)\psi'_l(\cdot - \alpha), \tag{2.3}$$

where $h_{jl}, \in \ell_0(\mathbb{Z}^s)$, for $j = 1, 2, \ldots, r, l = 1, 2, \ldots, n$. Denote $h_l = (h_{1l}, h_{2l}, \ldots, h_{rl})^T \in (\ell_0(\mathbb{Z}^s))^{r\times 1}, l = 1, 2, \ldots, n$. It is easy to check that $h_l \perp K(\psi)$ for $l = 1, 2, \ldots, n$. It follows from $K(\varphi) \subseteq K(\psi)$ that $h_l \perp K(\varphi) = V_\varphi^\perp$. By using of Lemma 2.1, we obtain $h_l \in V_\varphi$, for $l = 1, 2, \ldots, n$. Therefore, each h_l can be expressed as

$$h_l = \sum_{j=1}^{d} e_{jl} * c_j,$$

where $e_{jl} \in \ell_0(\mathbb{Z}^s)$, $j = 1, 2, \ldots, d$, $l = 1, 2, \ldots, n$. Since

$$\sum_{\alpha\in\mathbb{Z}^s} b(\alpha)\psi(\cdot - \alpha) = \sum_{\gamma\in\mathbb{Z}^s} (b * h)(\gamma)\psi'(\cdot - \gamma),$$

where $h = (h_{jl})_{1\leqslant j\leqslant r, 1\leqslant l\leqslant n} \in (\ell_0(\mathbb{Z}^s))^{r\times n}$, and h_l is the lth column of h. Therefore, there exists a constant C_2 such that

$$\left\| \sum_{\alpha\in\mathbb{Z}^s} b(\alpha)\psi(\cdot - \alpha) \right\|_p \leqslant C_2 \sum_{l=1}^{n} \|b * h_l\|_p. \tag{2.4}$$

Note that $b * h_l = \sum_{j=1}^{d} e_{jl} * (b * c_j)$. Hence

$$\|c * d_l\|_p = \left\| \sum_{j=1}^{d} e_{jl} * (b * c_j) \right\|_p \leqslant \sum_{j=1}^{d} \|e_{jl}\|_1 \|b * c_j\|_p, \quad l = 1, 2, \ldots, n. \tag{2.5}$$

We prove this lemma by using of (2.2), (2.4) and (2.5). □

After submitting this paper, I received a preprint of Q. Sun entitled “Linear dependence of the shifts of vector-valued compactly supported distributions”, in which Lemma 2.2 is proved with a different method (see [30]).

We are in a position to give characterizations of convergence of cascade algorithms in $(L_p(\mathbb{R}^s))^r$ $(1 \leqslant p \leqslant \infty)$.

Theorem 2.3. *Let* $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ *be the normalized solution of the refinement equation* (1.1) *in* $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$. *If* ψ *is a vector of compactly supported functions satisfying Strang-Fix conditions of order* 1, *and if* $K(\varphi) \subseteq K(\psi)$, *then*

$$\lim_{n\to\infty} \left\| Q_a^n \psi - \varphi \right\|_p = 0.$$

In the case $p = \infty$, *if both* φ *and* ψ *are continuous, then* $Q_a^n \psi$ *converges to* φ *uniformly.*

Proof of Theorem 2.3. We only consider the case $1 \leqslant p < \infty$. Since the proof of $p = \infty$ is similar. We pick g_j, $j = 1, 2, \ldots, r$ to be compactly supported functions in $L_{p'}(\mathbb{R}^s)$ such that $\int_{\mathbb{R}^s} g_1(x)\,dx = 1$ and $\int_{\mathbb{R}^s} g_j(x)\,dx = 0$, $j = 2, \ldots, r$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Let P_φ be the quasi-projection operator given by

$$P_\varphi f := \sum_{\alpha\in\mathbb{Z}^s} \sum_{j=1}^{r} < f, g_j(\cdot - \alpha) > \varphi_j(\cdot - \alpha), \quad f \in L_p(\mathbb{R}^s),$$

where $< f, g_j(\cdot - \alpha >$ denotes the inner product of two functions f and $g_j(\cdot - \alpha)$.

Since φ is the normalized solution of Eq. (1.1) in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$, it was proved in [21] that φ must satisfy Strang-Fix conditions of order 1 which imply that $e_1^T \hat{\varphi}(0) = 1$ and $e_1^T \hat{\varphi}(2\pi\beta) = 0$, for all $\beta \in \mathbb{Z}^s \backslash \{0\}$. By Poisson summation formula, one can easily prove that these conditions are equivalent to the condition $e_1^T \sum_{\alpha\in\mathbb{Z}^s} \varphi(\cdot - \alpha) = 1$. Therefore, quasi-projection operators P_φ reproduces constants, i.e., $P_\varphi 1 = 1$. Similarly, let P_ψ be the quasi-projection operator given by

$$P_\psi f := \sum_{\alpha\in\mathbb{Z}^s} \sum_{j=1}^{r} < f, g_j(\cdot - \alpha) > \psi_j(\cdot - \alpha), \quad f \in L_p(\mathbb{R}^s).$$

Since ψ also satisfy Strang-Fix conditions of order 1, then quasi-projection operator P_ψ also reproduces constants, i.e., $P_\psi 1 = 1$. Consequently, for $f \in L_p(\mathbb{R}^s)$, $1 \leqslant p \leqslant \infty$, we have (see [13])

$$\lim_{n\to\infty} \left\| f - \sum_{\alpha\in\mathbb{Z}^s} \sum_{j=1}^{r} < f, m^n g_j(M^n \cdot -\alpha) > \varphi_j(M^n \cdot -\alpha) \right\|_p = 0 \tag{2.6}$$

and

$$\lim_{n\to\infty} \left\| f - \sum_{\alpha\in\mathbb{Z}^s} \sum_{j=1}^{r} < f, m^n g_j(M^n \cdot -\alpha) > \psi_j(M^n \cdot -\alpha) \right\|_p = 0. \tag{2.7}$$

For $n = 1, 2, \ldots$, let c_n be the sequence of $r \times r$ matrices given by

$$c_n(\alpha) := \left(< \varphi_j, m^n g_l(M^n \cdot -\alpha) > \right)_{1 \leqslant j,l \leqslant r}, \quad \alpha \in \mathbb{Z}^s.$$

It follows from (2.6) and (2.7) that

$$\lim_{n\to\infty} \left\| \varphi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\varphi(M^n \cdot -\alpha) \right\|_p = 0, \tag{2.8}$$

and

$$\lim_{n\to\infty} \left\| \varphi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n \cdot -\alpha) \right\|_p = 0. \tag{2.9}$$

Since φ is the normalized solution of Eq. (1.1), we have

$$\varphi = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\varphi(M^n \cdot -\alpha),$$

where the sequence a_n ($n = 1, 2, \ldots$) are given in (1.10).

It follows that

$$\lim_{n\to\infty} \left\| \sum_{\alpha\in\mathbb{Z}^s} (c_n(\alpha) - a_n(\alpha))\varphi(M^n \cdot -\alpha) \right\|_p = 0.$$

Since $K(\varphi) \subseteq K(\psi)$, Lemma 2.2 tells us that there exists a positive constant C independent of n such that

$$\left\| \sum_{\alpha \in \mathbb{Z}^s} (c_n(\alpha) - a_n(\alpha))\psi(M^n \cdot - \alpha) \right\|_p \leqslant C \left\| \sum_{\alpha \in \mathbb{Z}^s} (c_n(\alpha) - a_n(\alpha))\varphi(M^n \cdot - \alpha) \right\|_p = C \left\| \sum_{\alpha \in \mathbb{Z}^s} c_n(\alpha)\varphi(M^n \cdot - \alpha) - \varphi \right\|_p .$$

But

$$\begin{aligned} \| Q_a^n \psi - \varphi \|_p &= \left\| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)\psi(M^n \cdot - \alpha) - \varphi \right\|_p \\ &\leqslant \left\| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)\psi(M^n \cdot - \alpha) - \sum_{\alpha \in \mathbb{Z}^s} c_n(\alpha)\psi(M^n \cdot - \alpha) \right\|_p \\ &\quad + \left\| \sum_{\alpha \in \mathbb{Z}^s} c_n(\alpha)\psi(M^n \cdot - \alpha) - \varphi \right\|_p \\ &\leqslant C \left\| \sum_{\alpha \in \mathbb{Z}^s} c_n(\alpha)\varphi(M^n \cdot - \alpha) - \varphi \right\|_p + \left\| \sum_{\alpha \in \mathbb{Z}^s} c_n(\alpha)\psi(M^n \cdot - \alpha) - \varphi \right\|_p . \end{aligned}$$

It follows from (2.8) and (2.9) that $\lim_{n\to\infty} \| Q_a^n \psi - \varphi \|_p = 0$, as desired. □

Remark 2.4. We remark that if the normalized solution φ of refinement equation (1.1) lies in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$, then it was proved in [21] that φ satisfies Strang-Fix conditions of order 1, i.e. $e_1^T \hat{\varphi}(0) = 1$, and $e_1^T \hat{\varphi}(2\beta\pi) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$. It was also proved in [21] that if ψ is a vector of compactly supported functions in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$, such that cascade algorithm with initial vector of function ψ converges to φ in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$, then ψ satisfies Strang-Fix conditions of order 1. Therefore, the assumption on ψ to satisfy Strang-Fix conditions of order 1 in Theorem 2.3 is necessary. It follows from Theorem 2.3 that if the shifts of $\varphi_1, \ldots, \varphi_r$ are linearly independent, then $K(\varphi) = \{0\}$ which implies that cascade algorithm $Q_a^n \psi$ converges to φ in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ for any vector of compactly supported functions $\psi \in (L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ satisfying Strang-Fix conditions of order 1. In fact, we can prove that if the shifts of $\varphi_1, \ldots, \varphi_r$ are stable, then cascade algorithm $Q_a^n \psi$ converges to φ in $(L_p(\mathbb{R}^s))(1 \leqslant p \leqslant \infty)$ for any vector of compactly supported functions $\psi \in (L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ satisfying Strang-Fix conditions of order 1.

Theorem 2.5. *Suppose $\varphi_0 = (\varphi_{1,0}, \ldots, \varphi_{r,0})^T$ is the vector of compactly supported functions in $(L_p(\mathbb{R}^s))^r$ for $1 \leqslant p \leqslant \infty$, φ_0 satisfies Strang-Fix conditions of order* 1. *If the shifts of $\varphi_{1,0}, \ldots, \varphi_{r,0}$ are stable. Let φ be the normalized solution of refinement equation* (1.1) *in $(L_p(\mathbb{R}^s))^r$ for $1 \leqslant p \leqslant \infty$ such that*

$$\lim_{n\to\infty} \| Q_a^n \varphi_0 - \varphi \|_p = 0,$$

then for any vector of compactly supported functions ψ *in* $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ *satisfying Strang-Fix conditions of order* 1, *we have*

$$\lim_{n\to\infty} \|Q_a^n \psi - \varphi\|_p = 0.$$

In particular, if the normalized solution $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ *of refinement equation* (1.1) *lies in* $(L_p(\mathbb{R}^s))^r$ *for* $1 \leqslant p \leqslant \infty$ *and the shifts of* $\varphi_1, \ldots, \varphi_r$ *are stable, then the cascade algorithm* $Q_a^n \psi$ *converges to* φ *in* $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ *for any vector of compactly supported functions* $\psi \in (L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ *satisfying Strang-Fix conditions of order* 1.

Proof of Theorem 2.5. The proof follows the lines of the proof of Theorem 2.3. We pick a vector of compactly supported functions in $(L_p(\mathbb{R}^s))^r$ satisfying Strang-Fix conditions of order 1. Let P_{φ_0} and P_ψ be the quasi-projection operators denoted as in the proof of Theorem 2.3. Since φ_0 and ψ satisfy Strang-Fix conditions of order 1. It follows from the proof of Theorem 2.3 that

$$\lim_{n\to\infty} \left\| \varphi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\varphi_0(M^n \cdot -\alpha) \right\|_p = 0$$

and

$$\lim_{n\to\infty} \left\| \varphi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n \cdot -\alpha) \right\|_p = 0,$$

where $(c_n(\alpha))$ is the sequence of $r \times r$ matrix as in the proof of Theorem 2.3. Since the shifts of $\varphi_{1,0}, \ldots, \varphi_{r,0}$ are stable, there exists a constant $C_2 > 0$ such that

$$\|a_n - c_n\|_p \leqslant C_2 m^{n/p} \left\| \sum_{\alpha\in\mathbb{Z}^s} (c_n(\alpha) - a_n(\alpha))\varphi_0(M^n \cdot -\alpha) \right\|_p ,$$

where a_n is defined by (1.10). Furthermore,

$$Q_a^n \psi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n \cdot -\alpha) = \sum_{\alpha\in\mathbb{Z}^s} (a_n(\alpha) - c_n(\alpha))\psi(M^n \cdot -\alpha).$$

Hence there exists a constant $C_3 > 0$ such that

$$\left\| Q_a^n \psi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n \cdot -\alpha) \right\|_p \leqslant C_3 m^{-n/p} \|a_n - c_n\|_p .$$

Combining the above estimate, we see that

$$\left\| Q_a^n \psi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n \cdot -\alpha) \right\|_p \leqslant C_2 C_3 \left\| \sum_{\alpha\in\mathbb{Z}^s} (a_n(\alpha) - c_n(\alpha))\varphi_0(M^n \cdot -\alpha) \right\|_p .$$

Therefore, we have

$$
\begin{aligned}
&\left\| Q_a^n\psi - \varphi \right\|_p \\
&\leqslant \left\| \varphi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n\cdot-\alpha) \right\|_p + \left\| Q_a^n\psi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n\cdot-\alpha) \right\|_p \\
&\leqslant \left\| \varphi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n\cdot-\alpha) \right\|_p + C_2C_3 \left\| \sum_{\alpha\in\mathbb{Z}^s} (a_n(\alpha)-c_n(\alpha))\varphi_0(M^n\cdot-\alpha) \right\|_p \\
&\leqslant \left\| \varphi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n\cdot-\alpha) \right\|_p + C_2C_3 \left\| \varphi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\varphi_0(M^n\cdot-\alpha) \right\|_p \\
&\quad + C_2C_3 \left\| \varphi - \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\varphi_0(M^n\cdot-\alpha) \right\|_p .
\end{aligned}
$$

It follows from the above discussions, we complete the proof of Theorem 2.5. □

Remark 2.6. We point out that Theorem 2.5 appeared in [6] for $r = 1$, in [27] for $s = 1, r = 1$ and $1 < p < \infty$ and in [17] for $s = 1$ and $M = 2$.

To obtain the estimates on convergence rate of cascade algorithm in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ with M being isotropic, we need some notations and some preliminary results. Since M is isotropic, then M is similar to a diagonal matrix $diag(\sigma_1, \ldots, \sigma_s)$ with $|\sigma_1| = \cdots = |\sigma_s| = m^{1/s}$, where $m = |det\, M|$. Therefore, there exists an invertible $s \times s$ matrix $\Lambda = (\lambda_{jl})_{1\leqslant j,l\leqslant s}$ such that

$$\Lambda M \Lambda^{-1} = diag(\sigma_1, \ldots, \sigma_s).$$

For $j = 1, 2, \ldots, s$, let q_j be the linear polynomial given by

$$q_j(x) = \sum_{l=1}^{s} \lambda_{jl} x_l, \qquad x = (x_1, \ldots, x_s) \in \mathbb{R}^s.$$

We use $q_j(D)$ to denote the differential operator $\sum_{l=1}^{s} \lambda_{jl} D_l$. For multi-index $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, define $q_\mu = q_1^{\mu_1}(x)\cdots q_s(x)^{\mu_s}$.

The factorial of a multi-index $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ is defined to be $\mu! := \mu_1! \cdots \mu_s!$. Let $\mu = (\mu_1, \ldots, \mu_s)$ and $\nu = (\nu_1, \ldots, \nu_s)$ be two multi-indexes. Then $\nu \leqslant \mu$ means $\nu_j \leqslant \mu_j$ for $j = 1, 2, \ldots, s$. By $\nu < \mu$ we mean $\nu \leqslant \mu$ and $\nu \neq \mu$. For $\nu \leqslant \mu$, define

$$\binom{\mu}{\nu} := \frac{\mu!}{\nu!(\mu-\nu)!}.$$

Let $B_\mu (|\mu| \leqslant k-1)$ be the $1 \times r$ vectors given by the recursive relation

$$B_\mu = \sum_{\nu\leqslant\mu} \binom{\mu}{\nu} \sigma^{\mu-\nu} B_{\mu-\nu} q_\nu(-iD) H(0), \tag{2.10}$$

where $\sigma^{\mu-\nu} = \sigma_1^{\mu_1-\nu_1} \cdots \sigma_s^{\mu_s-\nu_s}$, $B_0 = e_1^T$. This equation can be rewritten as

$$B_\mu(I_r - \sigma^\mu H(0)) = \sum_{0\neq\nu\leqslant\mu} \binom{\mu}{\nu} \sigma^{\mu-\nu} B_{\mu-\nu} q_\nu(-iD) H(0),$$

where I_r denotes the $r \times r$ identity matrix. Since matrix $H(0)$ satisfies Eigenvalue condition, then the matrix $I_r - \sigma^\mu H(0)$ is invertible for any multi-index μ with $0 < |\mu| \leqslant k-1$. Therefore, the vector $B_\mu (0 < |\mu| \leqslant k-1)$ are unique determined by (2.10). For $|\mu| \leqslant k-1$ and $f \in (W_p^{k-1}(\mathbb{R}^s))^r$, define $T_\mu f(\xi) = \sum_{\nu\leqslant\mu} \binom{\mu}{\nu} B_{\mu-\nu} q_\nu(-iD) \hat{f}(\xi)$.

With above notations, we introduce the following lemmas which were proved in [1].

Lemma 2.7 (*Chen et al. [1]*). *Suppose $\varphi_1, \ldots, \varphi_r$ are the compactly supported functions in $W_p^{k-1}(\mathbb{R}^s)$ for $1 \leqslant p \leqslant \infty$ such that the $r \times 1$ vector $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ is a normalized solution of the refinement equation* (1.1) *with M being isotropic. Let B_μ be denoted by* (2.10). *Then $T_\mu\varphi(0) = 0$ for all $0 < |\mu| \leqslant k-1$ and $T_\mu\varphi(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$ and all $|\mu| \leqslant k-1$.*

Lemma 2.7 tells us that if φ is compactly supported normalized solution of Eq. (1.1) in $(W_p^{k-1}(\mathbb{R}^s))^r$ for $1 \leqslant p \leqslant \infty$, then we have $T_\mu\varphi(0) = 0$ for all $0 < |\mu| \leqslant k-1$ and $T_\mu\varphi(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$ and all $|\mu| \leqslant k-1$. These conditions are called Strang-Fix conditions of order k (see [1,14,21]). By using the Poisson summation formula one can easily see that the Strang-Fix conditions of order k are equivalent to the following conditions:

$$\sum_{\alpha\in\mathbb{Z}^s} \sum_{\nu\leqslant\mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu B_{\mu-\nu} \varphi(x-\alpha) = (\Lambda x)^\mu \quad \forall |\mu| \leqslant k-1 \text{ a.e., } x \in \mathbb{R}^s. \tag{2.11}$$

Thus, φ reproduces all polynomials in Π_{k-1}, where Π_{k-1} denote the linear space of all polynomials of degree at most $k-1$.

Lemma 2.8 (*Chen et al. [1]*). *Suppose φ_0 is an $r \times 1$ vector of compactly supported functions in $(W_p^{k-1}(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ such that*

$$\lim_{n\to\infty} \left\| Q_a^n \varphi_0 - \varphi \right\|_{(W_p^{k-1}(\mathbb{R}^s))^r} = 0,$$

where $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ is a normalized solution of the refinement equation (1.1) *in $(W_p^{k-1}(\mathbb{R}^s))^r$, for $1 \leqslant p \leqslant \infty$, then $e_1^T \hat{\varphi}_0(0) = 1$ and $T_\mu\varphi_0(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$ and all $|\mu| \leqslant k-1$.*

By [1] we know that for each μ with $|\mu| \leqslant k-1$ there exists some $g_\mu \in \prod_{|\mu|-1}$ such that

$$\sum_{\alpha\in\mathbb{Z}^s} \sum_{\nu\leqslant\mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu B_{\mu-\nu} \varphi_0(x-\alpha) = (\Lambda x)^\mu + g_\mu(x), \quad x \in \mathbb{R}^s, \tag{2.12}$$

and $g_\mu = 0$ if and only if $T_\mu\varphi_0(0) = 0$ for all $1 \leqslant |\mu| \leqslant k-1$.

Following Theorem 2.9 gives a characterization of convergence rates of cascade algorithm in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$.

Theorem 2.9. *Let $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ be a normalized solution of the refinement equation* (1.1) *with dilation matrix M being isotropic. Suppose $\varphi \in (Lip(\eta, L_p(\mathbb{R}^s)))^r$ with $\eta > 0$ and*

$1 \leqslant p \leqslant \infty$. *Suppose k is the positive integer satisfying* $k-1 < \eta \leqslant k$. *Let* $\psi = (\psi_1, \ldots, \psi)^T$ *be in* $(L_p(\mathbb{R}^s))^r\,(1 \leqslant p \leqslant \infty)$ *satisfying Strang-Fix conditions of order k, i.e.,* ψ *satisfies* $T_\mu \psi(0) = 0$ *for all* $0 < |\mu| \leqslant k-1$ *and* $T_\mu \psi(2\pi\beta) = 0$ *for all* $\beta \in \mathbb{Z}^s \backslash \{0\}$ *and all* $|\mu| \leqslant k-1$. *If* $K(\varphi) \subseteq K(\psi)$, *then there exists a positive constant C such that*

$$\left\| Q_a^n \psi - \varphi \right\|_p \leqslant C (m^{-1/s})^{\eta n} \quad \forall n \in \mathbb{N}.$$

Proof of Theorem 2.9. Since the normalized solution φ of refinement equation (1.1) lies in $(Lip(\eta, L_p(\mathbb{R}^s)))^r$ with $k-1 < \eta \leqslant k$, then $\varphi \in (W_p^{k-1}(\mathbb{R}^s))^r$. By Lemma 2.7, φ satisfies Strang-Fix conditions of order k, therefore,

$$\sum_{\alpha \in \mathbb{Z}^s} \sum_{\nu \leqslant \mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu B_{\mu-\nu} \varphi(x-\alpha) = (\Lambda x)^\mu \quad \forall |\mu| \leqslant k-1 \text{ a.e.}, x \in \mathbb{R}^s.$$

For $B_{\mu-\nu} = (B^1_{\mu-\nu}, \ldots B^r_{\mu-\nu},)^T$, there exist real-valued compactly supported functions $g_1, \ldots, g_r \in L_{p'}(\mathbb{R}^s)$ such that

$$\int_{\mathbb{R}^s} (\Lambda x)^{\mu-\nu} g_j(x)\, dx = B^j_{\mu-\nu}, \quad j = 1, \ldots, r,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Let P_φ be the quasi-projection operator given by

$$P_\varphi f = \sum_{\alpha \in \mathbb{Z}^s} \sum_{j=1}^r < f, g_j(\cdot - \alpha) > \varphi_j(\cdot - \alpha), \quad f \in L_p(\mathbb{R}^s).$$

For $|\mu| \leqslant k-1$ we have

$$\begin{aligned} P_\varphi (\Lambda x)^\mu &= \sum_{\alpha \in \mathbb{Z}^s} \sum_{j=1}^r < (\Lambda x)^\mu, g_j(\cdot - \alpha) > \varphi_j(\cdot - \alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^s} \sum_{\nu \leqslant \mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu \int_{\mathbb{R}^s} (\Lambda x)^{\mu-\nu} g_j(x)\, dx\, \varphi_j(\cdot - \alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^s} \sum_{\nu \leqslant \mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu B^j_{\mu-\nu} \varphi_j(\cdot - \alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^s} \sum_{\nu \leqslant \mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu B_{\mu-\nu} \varphi(\cdot - \alpha) = (\Lambda x)^\mu. \end{aligned}$$

Thus, P_φ reproduces all polynomials of degree at most $k-1$, i.e., $P_\varphi q = q$ for all $q \in \Pi_{k-1}$. Let P_ψ be the quasi-projection operator given by

$$P_\psi f = \sum_{\alpha \in \mathbb{Z}^s} \sum_{j=1}^r < f, g_j(\cdot - \alpha) > \psi_j(\cdot - \alpha), \quad f \in L_p(\mathbb{R}^s).$$

By our assumptions on ψ, we know that P_ψ also reproduces all polynomials of degree at most $k-1$, i.e., $P_\psi q = q$ for all $q \in \Pi_{k-1}$.

Let a_n and c_n $(n = 1, 2, \ldots)$ be the same sequence s as above. Since $\varphi \in (Lip(\eta, L_p(\mathbb{R}^s)))^r$, from Theorem 5.1 of [13] we have

$$\left\| \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\varphi(M^n\cdot-\alpha)-\varphi \right\|_p \leqslant C_4(m^{-1/s})^{n\eta}$$

and

$$\left\| \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n\cdot-\alpha)-\varphi \right\|_p \leqslant C_4(m^{-1/s})^{n\eta},$$

where C_4 is positive constant independent of n. Since φ is a solution of refinement equation (1.1), we have $\varphi = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\varphi(M^n\cdot-\alpha)$. Hence, it follows that

$$\left\| \sum_{\alpha\in\mathbb{Z}^s} (c_n(\alpha)-a_n(\alpha))\varphi(M^n\cdot-\alpha) \right\|_p \leqslant C_4(m^{-1/s})^{n\eta}.$$

Note that $K(\varphi) \subseteq K(\psi)$, it follows from Lemma 2.2 that there exists a positive constant C_5 independent of n such that

$$\left\| \sum_{\alpha\in\mathbb{Z}^s} (c_n(\alpha)-a_n(\alpha))\psi(M^n\cdot-\alpha) \right\|_p \leqslant C_5(m^{-1/s})^{n\eta}.$$

Hence,

$$\begin{aligned}
&\left\|\varphi - Q_a^n\psi\right\|_p \\
&\quad\leqslant \left\|\varphi-\sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\psi(M^n\cdot-\alpha)\right\|_p + \left\|\sum_{\alpha\in\mathbb{Z}^s}(c_n(\alpha)-a_n(\alpha))\psi(M^n\cdot-\alpha)\right\|_p \\
&\quad\leqslant (C_4+C_5)(m^{-1/s})^{n\eta}.
\end{aligned}$$

We complete the proof of Theorem 2.9. □

Remark 2.10. Under assumptions that initial vector of function ψ satisfies Strang-Fix conditions of order k, we give a characterization of convergence rates of cascade algorithm in $(L_p(\mathbb{R}^s))^r (1\leqslant p\leqslant\infty)$. It follows from Lemma 2.8 that the assumptions on ψ to satisfy $T_\mu\psi(2\pi\beta) = 0$ for all $\beta\in\mathbb{Z}^s\backslash\{0\}$ and all $|\mu|\leqslant k-1$ are necessary. When $r = 1$, the conditions $T_\mu\psi(2\pi\beta) = 0$ for all $\beta\in\mathbb{Z}^s\backslash\{0\}$ and all $|\mu|\leqslant k-1$ are reduced to $D^\mu\hat{\psi}(2\pi\beta) = 0$ for all $\beta\in\mathbb{Z}^s\backslash\{0\}$ and all $|\mu|\leqslant k-1$. We note that Theorem 2.9 appeared also in [29] for $M = 2$.

3. Characterization of L_p smoothness of a refinable function

In this section, we will investigate the smoothness of solution of refinement equations (1.1) with $M = qI_{s\times s}$, where $q\geqslant 2$ is an integer. We give a characterization for the smoothness of a refinable function in terms of the corresponding refinement mask a and dilation matrix $qI_{s\times s}$. The proof of our result is based on following theorem.

Theorem 3.1. *Let $v > 0$ and k be a positive integer. Let $q \geqslant 2$ be an integer. If $\varphi \in (Lip^*(v, L_p(\mathbb{R}^s)))^r$ and $k > v$, then there exists a constant $C_6 > 0$ such that*

$$\|\nabla^k_{q^{-n}e_j}\varphi\|_p \leqslant C_6 q^{-nv} \quad \forall 1 \leqslant j \leqslant s, \ \ n \in \mathbb{N}. \tag{3.1}$$

Conversely, if $\varphi \in (L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ satisfies the conditions in (3.1), *then $\varphi \in (Lip^*(v, L_p(\mathbb{R}^s)))^r$.*

Proof. We only need to prove this theorem for $r = 1$. We follow the line of [4]. The proof is based on following results from Ditzian [2,3]: for a function φ in $L_p(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$ (φ is continuous in the case $p = \infty$), φ lies in Lip*$(v, L_p(\mathbb{R}))$ for $v > 0$ if and only if, for some integer $k > v$, there exists a constant $C_6 > 0$ such that

$$\|\nabla^k_{q^{-n}}\varphi\|_p \leqslant C_6 q^{-nv} \quad \forall 1 \leqslant j \leqslant s, \ \ n \in \mathbb{N}. \tag{3.2}$$

If inequality (3.1) holds true for $r = 1$, we pick h to be a function such that $\|h\|_{p'} \leqslant 1$, where $\frac{1}{p} + \frac{1}{p}' = 1$. Define h_1 to be the convolution of φ and h, i.e.,

$$h_1(x) := \int_{\mathbb{R}^s} \varphi(x-t)h(t)\,dt, \quad x \in \mathbb{R}^s.$$

It is easily seen that h_1 is continuous and bounded. It follows from (3.1) that for $j = 1, 2, \ldots, s$,

$$\|\nabla^k_{q^{-n}e_j} h_1\|_\infty = \|(\nabla^k_{q^{-n}e_j}\varphi) * h\|_\infty \leqslant \|\nabla^k_{q^{-n}e_j}\varphi\|_p \|h\|_{p'} \leqslant C_6 q^{-nv} \quad \forall n \in \mathbb{N}.$$

Hence, we have

$$|\nabla^k_{q^{-n}e_j} h_1(te_j)| \leqslant C_6 q^{-nv} \quad \forall t \in \mathbb{R}, \ \ n \in \mathbb{N}.$$

By (3.2), we know that there exists a positive constant C_7 independent of h such that

$$|\nabla^k_{\tau e_j} h_1(te_j)| \leqslant C_7 \tau^v \quad \forall t \in \mathbb{R}, \ \ \tau > 0. \tag{3.3}$$

It follows from inequality (3.3) that for any function h with $\|h\|_{p'} \leqslant 1$, we have

$$|\nabla^k_{\tau e_j} h_1(0)| = |(\nabla^k_{\tau e_j}\varphi) * h(0)| \leqslant C_7 \tau^v, \quad j = 1, \ldots, s \quad \forall \tau > 0.$$

Therefore,

$$\|\nabla^k_{\tau e_j}\varphi\|_p = \sup_{\|h\|_{p'} \leqslant 1} \left| \int_{\mathbb{R}^s} (\nabla^k_{\tau e_j}\varphi)(-x)h(x)\,dx \right| \leqslant C_7 \tau^v, \quad j = 1, \ldots, s, \ \ \tau > 0.$$

Following [2], we know that $\varphi \in Lip^*(v, L_p(\mathbb{R}^s))$. If $\varphi \in Lip^*(v, L_p(\mathbb{R}^s))$, by the definition of Lipschitz space $Lip^*(v, L_p(\mathbb{R}))$, there exists a positive constant C_8 such that

$$\|\nabla^k_{q^{-n}e_j}\varphi\|_p \leqslant C_8 q^{-nv} \quad \forall 1 \leqslant j \leqslant s, \ \ n \in \mathbb{N}.$$

We complete the proof of Theorem 3.1. □

By using some methods similar to [11,18], we obtain

Lemma 3.2. *Let* $\varphi = (\varphi_1, \ldots, \varphi_r)^T \in (L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ *be a compactly supported solution of the refinement equation* (1.1) *with mask a and dilation matrix* $M = qI_{s\times s}$, *where* $q \geqslant 2$ *is an integer. Let* a_n *be defined by* (1.10). *If there exists a constant* C_9 *such that for* $j = 1, \ldots, s$

$$\|\nabla_j^k a_n\|_p \leqslant C_9 q^{(s/2-v)n} \quad \forall n \in \mathbb{N}, \tag{3.4}$$

then $\varphi \in (Lip^*(v, L_p(\mathbb{R}^s)))^r$. *Conversely, if* $\varphi \in (Lip^*(v, L_p(\mathbb{R}^s)))^r$ *for* $1 \leqslant p \leqslant \infty$ *and* $k > v$, *then* (3.4) *holds true, provided that the shifts of* $\varphi_1, \ldots, \varphi_r$ *are stable.*

We will use the p-norm joint spectral radius of a finite collection of some linear operators restricted to a certain finite dimensional common invariant subspace to characterize the critical exponent $v_p(\varphi)$ of a vector refinement function φ. Let us review some notations of p-norm joint spectral radius from [9]. Let $\mathcal{A}$ be a finite collection of some linear operators on $(\ell_0(\mathbb{Z}^s))^r$. For a positive integer n we denote by $\mathcal{A}^n$ the Cartesian power of $\mathcal{A}$:

$$\mathcal{A}^n = \{(A_1, \ldots, A_n) : A_1, \ldots, A_n \in \mathcal{A}\}.$$

When $n = 0$, we interpret $\mathcal{A}^0$ as the set $\{I\}$, where I is the identity mapping.

Let

$$\|\mathcal{A}^n\|_\infty := \max\{\|A_1 \cdots A_n\| : (A_1, \ldots, A_n) \in \mathcal{A}^n\}.$$

Then the uniform joint spectral radius of $\mathcal{A}$ is defined to be

$$\rho_\infty(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_\infty^{1/n}. \tag{3.5}$$

The p-norm joint spectral radius of $\mathcal{A}$ is defined to be

$$\rho_p(\mathcal{A}) = \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n}. \tag{3.6}$$

Let a be an element of $(\ell_0(\mathbb{Z}^s))^r \times r$. For $\varepsilon \in \{\gamma_j, j = 0, 1, \ldots, q^s - 1\}$, where $\{\gamma_j, j = 0, 1, \ldots, q^s - 1\}$ be the q^s distinct elements of coset spaces $\mathbb{Z}^s / q\mathbb{Z}^s$ with $\gamma_0 = 0$. We denote $E = \{\gamma_k, k = 0, 1, \ldots, q^s - 1\}$. Thus, each element $\alpha \in \mathbb{Z}^s$ can be uniquely represented as $\varepsilon + q\gamma$, where $\varepsilon \in E$ and $\gamma \in \mathbb{Z}^s$. For $\varepsilon \in E$, and $a \in (\ell_0(\mathbb{Z}^s))^{r\times r}$, we define the linear operators A_ε on $(\ell_0(\mathbb{Z}^s))^r$ as

$$A_\varepsilon u(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\varepsilon + q\alpha - \beta)u(\beta), \quad \alpha \in \mathbb{Z}^s, \ \ u \in (\ell_0(\mathbb{Z}^s))^r. \tag{3.7}$$

Following almost word for word the consideration of the proof of Lemma 3.3 in [18], we have

Lemma 3.3. *Let a be an element of* $(\ell_0(\mathbb{Z}^s))^{r\times r}$, *and let* $a_n (n = 1, 2, \ldots)$ *be given by* (1.10) *with* $M = qI_{s\times s}$, *where* $q \geqslant 2$ *is an integer. For* $\varepsilon \in E$, *let* A_ε *be the linear operator on* $(\ell_0(\mathbb{Z}^s))^r$ *defined by* (3.7). *Let* $\mathcal{A} = \{A_\varepsilon; \varepsilon \in E\}$. *Then, we have for* $k \geqslant 0$,

$$\lim_{n\to\infty} \|\nabla_j^k a_n\|_p^{1/n} = \rho_p(\mathcal{A}|_V),$$

where V *is the minimal common invariant subspace of* $\mathcal{A}$ *generated by* $e_l(\Delta_j^k \delta)$, $1 \leqslant j \leqslant s$, $1 \leqslant l \leqslant r$.

By using same method as the proof of Theorem 3.3 in [18], we can obtain main result of this section. The proof of theorem is omitted.

Theorem 3.4. *Let $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ be the compactly supported L_p-solution of Eq.* (1.1) *with $a \in (\ell_0(\mathbb{Z}^s))^{r\times r}$ and $M = qI_{s\times s}$, where $q \geqslant 2$ is an integer. For $\varepsilon \in E$, let A_ε be the linear operator on $(\ell_0(\mathbb{Z}^s))^r$ defined by* (3.7). *Let $\mathcal{A} = \{A_\varepsilon; \varepsilon \in E\}$, k be a positive integer and V be minimal common invariant subspace of $\mathcal{A}$ generated by $e_l(\Delta_j^k\delta)$, $1 \leqslant j \leqslant s$, $1 \leqslant l \leqslant r$. Then*

$$\nu_p(\varphi) \geqslant 1/p - \frac{1}{s} log_q \rho_p(\mathcal{A}|_V), \tag{3.8}$$

in addition, if the shifts of $\varphi_1, \ldots, \varphi_r$ are stable and if $k > 1/p - \frac{1}{s} log_q \rho_p(\mathcal{A}|_V)$, then equality (3.8) *holds*.

Remark 3.5. Theorem 3.4 was established in [18] for the case $s = 1$ and $M = 2$. In [15], Jia et al. gave a complete characterization for the smoothness of the L_p-solution of Eq. (1.1) without assuming stability when $s = 1$, $M = 2$ and $1 \leqslant p \leqslant \infty$. When $r = 1$ and $1 < p < \infty$, the smoothness of refinable distributions was characterized in [27,28] for $s = 1$ and also in [24,29] for $M = 2I_{s\times s}$.

Acknowledgments

The work was finished when the author visited University of Alberta, the author thanks Professor R.Q. Jia for his valuable suggestions and financial support. Thanks are also due to referees for some useful comments.

References

[1] D.R. Chen, R.Q. Jia, S.D. Riemenschneider, Convergence of vector subdivision schemes in Sobolev spaces, Appl. Comput. Harmon. Anal. 12 (2002) 128–149.
[2] Z. Ditzian, Moduli of continuity in $\mathbb{R}^s$ and $D \subset \mathbb{R}^s$, Trans. Amer. Math. Soc. 282 (1984) 611–623.
[3] Z. Ditzian, Moduli of smoothness using discrete data, J. Approx. Theory 49 (1987) 115–129.
[4] B. Han, Analysis and construction of optimal multivariate biorthogonal wavelets with compact support, SIAM J. Math. Anal. 2 (1999) 274–304.
[5] B. Han, The initial functions in a cascade algorithm, in: D.X. Zhou (Ed.), Proceeding of International Conference of Computational Harmonic Analysis in Hong Kong, 2000.
[6] B. Han, R.Q. Jia, Multivariate refinement equations and convergence of subdivision schemes, SIAM J. Math. Anal. 29 (1998) 1177–1199.
[7] C. Helli, D. Colella, Characterizations of scaling functions; continuous solutions, SIAM J. Matrix Anal. Appl. 15 (1994) 496–518.
[8] C. Heil, D. Colella, Matrix refinement equations: existence and uniqueness, J. Fourier Anal. Appl. 2 (1996) 363–377.
[9] R.Q. Jia, Subdivision schemes in L_p spaces, Adv. Comput. Math. 3 (1995) 309–341.
[10] R.Q. Jia, Stability of the shifts of a finite number of functions, J. Approx. Theory 95 (1998) 149–202.
[11] R.Q. Jia, Characterization of smoothness of multivariate refinable functions in Sobolev space, Trans. Amer. Math. Soc. 351 (1999) 4089–4112.
[12] R.Q. Jia, Convergence rates of cascade algorithms, Proc. Amer. Math. Soc 131 (2003) 1739–1749.
[13] R.Q. Jia, Approximation with scaled shift-invariant spaces by means of quasi-projection operator, J. Approx. Theory 131 (2004) 30–46.
[14] R.Q. Jia, Q.T. Jiang, Spectral analysis of the transition operator and its applications to smoothness analysis of wavelets, SIAM J. Matrix Anal. Appl. 24 (2003) 1071–1109.
[15] R.Q. Jia, K.S. Lau, D.X. Zhou, L_p solutions of refinement equations, J. Fourier Anal. Appl. 2 (2001) 143–169.

[16] R.Q. Jia, C.A. Micchelli, On linear independence of integer translates of a finite number of functions, Proc. Edinburgh Math. Soc. 36 (1992) 69–85.
[17] R.Q. Jia, S.D. Riemenschneider, D.X. Zhou, Vector subdivision schemes and multiple wavelets, Math. Comp. 67 (1998) 1533–1563.
[18] R.Q. Jia, S.D. Riemenschneider, D.X. Zhou, Smoothness of multiple refinable functions and multiple wavelets, SIAM J. Matrix Anal. Appl. 21 (1999) 1–28.
[19] Q.T. Jiang, Multivariate matrix refinement functions with arbitrary matrix dilation, Trans. Amer. Math. Soc. 351 (1999) 2407–2438.
[20] Q.T. Jiang, Z.W. Zhen, On existence and week stability of matrix refinable functions, Constr. Approx. 15 (1999) 337–353.
[21] S. Li, Vector subdivision schemes in $(L_p(\mathbb{R}^s))^r (1 \leqslant p \leqslant \infty)$ spaces, Sci. China (Ser. A) 46 (3) (2003) 364–375.
[22] S. Li, Characterization of smoothness of multivariate refinable functions and convergence of cascade algorithms associated with nonhomogeneous refinement equations, Adv. Comput. Math. 20 (2004) 311–331.
[23] K.S. Lau, J.R. Wang, Characterization of L^p-solutions for two-scale dilation equations, SIAM J. Math. Anal. 26 (1995) 1018–1046.
[24] B. Ma, Q. Sun, Compactly supported refinable distributions in Triebel–Lizorkin space and Besov space, J. Fourier Anal. Appl. 5 (1999) 87–104.
[25] C.A. Micchelli, H. Prautzsch, Uniform refinement of curves, Linear Algebra Appl. 114/115 (1989) 841–870.
[26] A. Ron, Wavelets and their associated operators, in: C.K. Chui, L.L. Schumaker (Eds.), Approximation Theory, vol. IX, Vanderbilt University, New York, 1998, pp. 283–317.
[27] Q. Sun, Convergence and boundedness of cascade algorithm in Besov space and Triebel–Lizorkin space I, Adv. Math. (China) 29 (2000) 507–526.
[28] Q. Sun, Convergence and boundedness of cascade algorithm in Besov space and Triebel–Lizorkin space II, Adv. Math. (China) 30 (2001) 22–36.
[29] Q. Sun, Convergence of cascade algorithms and smoothness of refinable distributions, Chinese Ann. Math. Ser. B 24 (2003) 367–386.
[30] Q. Sun, Linear dependence of the shifts of vector-valued compactly supported distributions, manuscript.